

WEAK INVARIANCE PRINCIPLE IN BESOV SPACES FOR STATIONARY MARTINGALE DIFFERENCES

DAVIDE GIRAUDO AND ALFREDAS RAČKAUSKAS

ABSTRACT. The classical Donsker weak invariance principle is extended to a Besov spaces framework. Polygonal line processes build from partial sums of stationary martingale differences as well independent and identically distributed random variables are considered. The results obtained are shown to be optimal.

Keywords: invariance principle, martingale differences, stationary sequences, Besov spaces.

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1. INTRODUCTION AND MAIN RESULTS

By weak invariance principle in a topological function space, say, E we understand the weak convergence of a sequence of probability measures induced on E by normalized polygonal line processes build from partial sums of random variables. The choice of the space E is important due to possible statistical applications via continuous mappings. Since stronger topology generates more continuous functionals, it is beneficial to have the weak invariance principle proved in as strong as possible topological framework.

Classical Donsker's weak invariance principle considers the space $E = C[0, 1]$ and polygonal line processes build from partial sums of i.i.d. centred random variables with finite second moment. An intensive research has been done in order to extend Donsker's result to a stronger topological framework as well to a larger class of random variables (see, e.g., [9], [12], [4] and references therein).

In this paper we consider weak invariance principle in Besov spaces for a class of strictly stationary sequence of martingale differences. To be more precise, let us first introduce some notation and definitions used throughout the paper.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $T: \Omega \rightarrow \Omega$ be a bijective bi-measurable transformation preserving the probability \mathbb{P} . The quadruple $(\Omega, \mathcal{F}, \mathbb{P}, T)$ is referred to as dynamical system (see, e.g., [10] for some background material). We assume that there is a sub- σ -algebra $\mathcal{F}_0 \subset \mathcal{F}$ such that $T\mathcal{F}_0 \subset \mathcal{F}_0$ and by \mathcal{I} we denote the σ -algebra of the sets $A \in \mathcal{F}$ such that $T^{-1}A = A$.

Next we consider a strictly stationary sequence $(X_j, j \geq 0)$ constructed as $X_j := f \circ T^j$, where $f: \Omega \rightarrow \mathbb{R}$ is \mathcal{F}_0 -measurable. We define also a non-decreasing filtration $\mathcal{F}_k = T^{-k}\mathcal{F}_0, k \geq 1$. Note that $(X_j, \mathcal{F}_j, j \geq 0)$ is then a martingale differences sequence provided $\mathbb{E}(f | T\mathcal{F}_0) = 0$.

Set

$$S_{f,0} := 0, \quad S_{f,n} := \sum_{j=0}^{n-1} f \circ T^j, \quad n \geq 1.$$

Our main object of investigation is the sequence of polygonal line processes $\zeta_{f,n} := (\zeta_{f,n}(t), t \in [0, 1])$, $n \geq 1$, defined by

$$\zeta_{f,n}(t) := S_{f, \lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)f \circ \mathbb{T}^{\lfloor nt \rfloor},$$

where for a real number $a \geq 0$, $\lfloor a \rfloor := \max\{k : k \in \{0, 1, \dots\}, k \leq a\}$. To define the paths space under consideration let $L_p([0, 1])$ be the space of Lebesgue integrable functions with exponent p ($1 \leq p < \infty$) and the norm

$$\|x\|_p = \left(\int_0^1 |x(t)|^p dt \right)^{1/p}, \quad x \in L_p([0, 1]).$$

If $x \in L_p([0, 1])$ its L_p -modulus of smoothness is defined as

$$\omega_p(x, \delta) = \sup_{|h| \leq \delta} \left(\int_{I_h} |x(t+h) - x(t)|^p dt \right)^{1/p}, \quad \delta \in [0, 1],$$

where $I_h = [0, 1] \cap [-h, 1-h]$. Let $\alpha \in [0, 1)$. The Besov space $B_{p,\alpha}^o = B_{p,\alpha}^o([0, 1])$ is defined by

$$B_{p,\alpha}^o := \left\{ x \in L_p([0, 1]) : \lim_{\delta \rightarrow 0} \delta^{-\alpha} \omega_p(x, \delta) = 0 \right\}.$$

Endowed with the norm

$$\|x\|_{p,\alpha} = \|x\|_p + \sup_{\delta \in (0,1)} \delta^{-\alpha} \omega_p(x, \delta), \quad x \in B_{p,\alpha}^o,$$

the space $B_{p,\alpha}^o$ is a separable Banach space and the following embeddings are continuous:

$$\begin{aligned} B_{p,\alpha}^o &\hookrightarrow B_{p,\beta}^o, \quad \text{for } 0 \leq \beta < \alpha; \\ B_{p,\alpha}^o &\hookrightarrow B_{q,\alpha}^o, \quad \text{for } 1 \leq q < p < \infty. \end{aligned}$$

Each $B_{p,\alpha}^o(0, 1)$ where $p \geq 1, 0 \leq \alpha < 1/2$, supports a standard Wiener process $W = (W(t), 0 \leq t \leq 1)$ (see, e.g., [14]). We note also, that any polygonal line process belongs to each of $B_{p,\alpha}^o$, $p \geq 1, \alpha \in [0, 1)$.

As usually $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.

Theorem 1.1. *Let $p > 2$, $1/p < \alpha < 1/2$ and*

$$q(p, \alpha) := 1/(1/2 - \alpha + 1/p). \tag{1.0.1}$$

Let $(f \circ \mathbb{T}^i, \mathbb{T}^{-i}\mathcal{F}_0, i \geq 0)$ be a martingale differences sequence. Assume that the following two conditions hold :

- (i) $\lim_{t \rightarrow \infty} t^{q(p,\alpha)} \mathbb{P}\{|f| \geq t\} = 0$;
- (ii) $\mathbb{E} \left([\mathbb{E}(f^2 | \mathbb{T}\mathcal{F}_0)]^{q(p,\alpha)/2} \right) < \infty$.

Then the convergence $n^{-1/2} \zeta_{f,n} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \sqrt{\mathbb{E}(f^2 | \mathcal{I})} W$ holds in the space $B_{p,\alpha}^o$, where W is independent of $\mathbb{E}(f^2 | \mathcal{I})$.

Let us note that condition (i) is stronger for the function f than its square integrability since (i) yields $\mathbb{E}(|f|^r) < \infty$ for any $r < q(p, \alpha)$ and $q(p, \alpha) > 2$ when $\alpha > 1/p$. However as shows our next result, condition (i) is necessary and sufficient for independent identically distributed

(i.i.d.) random variables. To formulate the result let Y, Y_1, Y_2, \dots be mean zero i.i.d. random variables with finite variance $\sigma^2 = \mathbb{E}(Y^2) > 0$. Let $\xi_n = (\xi_n(t), t \in [0, 1])$, be defined by

$$\xi_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} X_k + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}.$$

Theorem 1.2. *Let $p > 2, 1/p < \alpha < 1/2$ and let $q(p, \alpha)$ be defined by (1.0.1). Then*

$$n^{-1/2} \sigma^{-1} \xi_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} W \quad \text{in the space } B_{p, \alpha}^o \quad (1.0.2)$$

if and only if

$$\lim_{t \rightarrow \infty} t^{q(p, \alpha)} \mathbb{P}\{|Y| \geq t\} = 0. \quad (1.0.3)$$

Since $B_{\infty, \alpha}^o$ matches Hölder spaces we see, that Theorems 1.2 and 1.1 complement the weak invariance principle obtained in a Hölderian framework by [12] and [4]. Concerning condition (ii) of Theorem 1.1 we prove a need of certain extra assumption by a counterexample which for any dynamical system with positive entropy constructs a function f that satisfies the condition (i) but the convergence of polygonal line processes fails. Precise result reads as follows.

Theorem 1.3. *Let $p > 2, 1/p < \alpha < 1/2$ and $q(p, \alpha)$ be given by (1.0.1). For each dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T})$ with positive entropy, there exists a function $m: \Omega \rightarrow \mathbb{R}$ and a σ -algebra \mathcal{F}_0 for which $\mathbb{T}\mathcal{F}_0 \subset \mathcal{F}_0$ such that:*

- (i) *the sequence $(m \circ \mathbb{T}^i, \mathbb{T}^{-i}\mathcal{F}_0, i \geq 0)$ is a martingale difference sequence;*
- (ii) *the convergence $\lim_{t \rightarrow +\infty} t^{q(p, \alpha)} \mathbb{P}(|m| \geq t) = 0$ takes place;*
- (iii) *the sequence $(n^{-1/2} \zeta_{m, n})$ is not tight in $B_{p, \alpha}^o$.*

As it is seen from our next results the case where $0 \leq \alpha \leq 1/p$ is indeed quite different from the previously considered case where $1/p < \alpha < 1/2$.

Theorem 1.4. *Let $p \geq 1$ and $\alpha \in [0, 1/2) \cap [0, 1/p]$. Let $(f \circ \mathbb{T}^i, \mathbb{T}^{-i}\mathcal{F}_0, i \geq 0)$ be a martingale differences sequence. If $\mathbb{E} f^2 < \infty$ then $n^{-1/2} \zeta_{f, n} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \sqrt{\mathbb{E}(f^2 | \mathcal{I})} W$ in the space $B_{p, \alpha}^o$, where W is independent of $\mathbb{E}(f^2 | \mathcal{I})$.*

Theorem 1.5. *Let α and p be as in Theorem 1.4. Then*

$$n^{-1/2} \sigma^{-1} \xi_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} W \quad \text{in the space } B_{p, \alpha}^o. \quad (1.0.4)$$

Let us note that the finiteness of the second moment $\mathbb{E} Y^2$ is necessary for the convergence (1.0.4).

The rest of the paper is organized as follows. In Section 2 we shortly present needed information on structure of Besov spaces and tightness of measures on these spaces. Section 3 contains proofs of Theorems 1.1, 1.2 and 1.3 whereas Section 4 is devoted to the proofs of Theorems 1.4 and 1.5. Finally, in Section 6 we discuss possible applications of invariance principle in the Besov framework.

2. SOME FUNCTIONAL ANALYSIS AND PROBABILISTIC TOOLS

We denote by \mathcal{D}_j the set of dyadic numbers in $[0, 1]$ of level j , i.e.

$$\mathcal{D}_0 = \{0, 1\}, \quad \mathcal{D}_j = \{(2l-1)2^{-j}; 1 \leq l \leq 2^{j-1}\}, \quad j \geq 1.$$

Set

$$\mathcal{D} = \bigcup_{j \geq 0} \mathcal{D}_j$$

and write for $r \in \mathcal{D}_j$,

$$r^- := r - 2^{-j}, \quad r^+ := r + 2^{-j}.$$

The triangular Faber-Schauder functions Λ_r for $r \in \mathcal{D}_j$, $j > 0$, are

$$\Lambda_r(t) = \begin{cases} 2^j(t - r^-) & \text{if } t \in (r^-, r]; \\ 2^j(r^+ - t) & \text{if } t \in (r, r^+]; \\ 0 & \text{else.} \end{cases}$$

When $j = 0$, we just take the restriction to $[0, 1]$ in the above formula, so

$$\Lambda_0(t) = 1 - t, \quad \Lambda_1(t) = t, \quad t \in [0, 1].$$

Theorem 2.1 ([3]). *Let $p > 1$ and $1/p < \alpha < 1$. The Faber-Schauder system $\{\Lambda_r, r \in \mathcal{D}\}$ is the Schauder basis for $B_{p,\alpha}^o$: each $x \in B_{p,\alpha}^o$ has the unique representation,*

$$x = \sum_{r \in \mathcal{D}} \lambda_r(x) \Lambda_r,$$

where

$$\lambda_r(x) := x(r) - \frac{x(r^+) + x(r^-)}{2}, \quad r \in \mathcal{D}_j, \quad j \geq 1$$

and in the special case $j = 0$,

$$\lambda_0(x) := x(0), \quad \lambda_1(x) := x(1).$$

Moreover the norm is equivalent to the sequential norm:

$$\|x\|_{p,\alpha} \sim \|x\|_{p,\alpha}^{\text{seq}} := \sup_{j \geq 0} 2^{j\alpha - i/p} \left(\sum_{r \in \mathcal{D}_j} |\lambda_r(x)|^p \right)^{1/p}.$$

The Schmidt orthogonalization procedure (with respect to inner product in $L_2(0, 1)$) applied to Faber-Schauder system leads to the Franklin system $\{f_k, k \geq 0\}$:

$$f_k(t) = \sum_{i=0}^k c_{ik} \Lambda_i(t), \quad t \in [0, 1],$$

with $c_{kk} > 0$ for $k \geq 0$, where the matrix (c_{ik}) is uniquely determined.

Theorem 2.2 ([3]). *The Franklin system $\{f_n, n \geq 0\}$ is the basis for $B_{p,\alpha}^o$, $p \geq 1, 0 \leq \alpha < 1$: each $x \in B_{p,\alpha}^o$ has the unique representation,*

$$x = \sum_{k=0}^{\infty} x_k f_k,$$

where $x_k = \langle x, f_k \rangle := \int_0^1 x(t) f_k(t) dt$, $k \geq 0$.

The following proposition is proved in [2] for $\alpha > 1/p$ but similar arguments works as well for any $0 \leq \alpha < 1$.

Proposition 2.3. *Let $p \geq 1$ and $0 \leq \alpha < 1$. The set $K \subset B_{p,\alpha}^o$ is relatively compact if and only if*

- (i) $\sup_{x \in K} \|x\|_p < \infty$,
- (ii) $\lim_{\delta \rightarrow 0} \sup_{x \in K} \delta^{-\alpha} \omega_{p,\alpha}(x, \delta) = 0$.

Proof. One easily checks that (i) and (ii) yields relative compactness of K in $L_p([0, 1])$. Therefore for any sequence $(x_n)_{n \geq 1}$ of K there exists a subsequence, which we denote also (x_n) , converging in $L_p([0, 1])$ to some $x \in L_p([0, 1])$. To finish the proof it suffices to prove that

- (a) $x \in B_{p,\alpha}^o$;
- (b) $(x_n)_{n \geq 1}$ is a Cauchy sequence in $B_{p,\alpha}^o$.

Taking a.s. convergence subsequence $(x_{n'})$ and applying Fatou lemma we easily obtain for any $0 < \delta \leq 1$,

$$\omega_p(x, \delta) \leq \liminf_{n'} \omega_p(x_{n'}, \delta) \leq \sup_n \omega_p(x_n, \delta). \quad (2.0.1)$$

This yields (a). To prove (b) observe that for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $\delta^{-\alpha} \sup_n \omega_p(x_n, \delta) < \varepsilon$ when $\delta < \delta_\varepsilon$, hence, for $n, m \geq 1$

$$\begin{aligned} \|x_n - x_m\|_{p,\alpha} &= \|x_n - x_m\|_p + \max \left\{ \left[\sup_{0 < \delta \leq \delta_\varepsilon} ; \sup_{\delta_\varepsilon < \delta \leq 1} \right] \delta^{-\alpha} \omega_p(x_n - x_m, \delta) \right\} \\ &\leq \|x_n - x_m\|_p + \varepsilon + 2\delta_\varepsilon^{-\alpha} \|x_n - x_m\|_p \end{aligned}$$

and we complete the proof since $\lim_{n,m \rightarrow \infty} \|x_n - x_m\|_p = 0$. \square

Consider stochastic processes $Z, (Z_n)_{n \geq 1}$ with paths space $B_{p,\alpha}^o$ which is endowed with Borel σ -algebra $\mathcal{B}(B_{p,\alpha}^o)$. Let $P_Z, P_{Z_n}, n \geq 1$, be the corresponding distributions. As generally accepted the sequence (Z_n) converges in distribution to Z in $B_{p,\alpha}^o$ (denoted $Z_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Z$ in $B_{p,\alpha}^o$) provided (P_{Z_n}) converges weakly to P_Z : $\lim_{n \rightarrow \infty} \int f d P_{Z_n} = \int f d P_Z$ for each bounded continuous $f : B_{p,\alpha}^o \rightarrow \mathbb{R}$. The sequence (Z_n) is tight in $B_{p,\alpha}^o$ if for each $\varepsilon > 0$ there is a relatively compact set $K_\varepsilon \subset B_{p,\alpha}^o$ such that $\inf_{n \geq 1} P(Z_n \in K_\varepsilon) > 1 - \varepsilon$. Due to the well known Prohorov's theorem convergence in distribution in a separable metric space is coherent with tightness. Indeed, to prove convergence in distribution one has to establish tightness and to ensure uniqueness of the limiting distributions.

The following tightness criterion is obtained from Proposition 2.3.

Theorem 2.4. *The sequence (Z_n) of random processes with paths in $B_{p,\alpha}^o(0, 1)$ is tight if and only if the following two conditions are satisfied:*

- (i) $\lim_{b \rightarrow \infty} \sup_{n \geq 1} P(\|Z_n\|_p > b) = 0$;
- (ii) for each $\varepsilon > 0$

$$\lim_{\delta \rightarrow 0} \sup_{n \geq 1} P(\omega_{p,\alpha}(Z_n, \delta) \geq \varepsilon) = 0.$$

Proof. See, e.g., the proof of Theorem 8.2. in [1]. \square

Theorem 2.5. *Let $1 > \alpha > 1/p$. The sequence (Z_n) of random elements in the Besov space $B_{p,\alpha}^o(0, 1)$ is tight if and only if the following two conditions are satisfied:*

- (i) $\lim_{b \rightarrow \infty} \sup_n \mathbb{P}\{\|Z_n\|_p > b\} = 0;$
- (ii) for each $\varepsilon > 0$, $\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\sup_{j \geq J} 2^{j\alpha-j/p} \left(\sum_{r \in D_j} |\lambda_r(\zeta_n)|^p\right)^{1/p} > \varepsilon\right) = 0.$

Proof. It is just a corollary of tightness criterion established in [15] for Schauder decomposable Banach spaces as Besov spaces $B_{p,\alpha}^o$ with $\alpha > 1/p$, are such. \square

3. PROOFS: THE CASE $\alpha > 1/p$

We start this section with some auxiliary results which could be helpful when dealing with weak invariance principle for stationary sequences. Throughout we denote

$$W_{f,n} = n^{-1/2} \zeta_n.$$

3.1. Auxiliary results.

Lemma 3.1. *Let $p \geq 1$ and $\alpha > 1/p$. Assume that Z is a random element in both spaces $C[0,1]$ and $B_{p,\alpha}^o$. Then for any stationary sequence $(f \circ T^j)$ if*

- (i) $W_{f,n} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Z$ in $C[0,1]$, and
- (ii) $(W_{f,n})$ is tight in $B_{p,\alpha}^o$,

then $W_{f,n} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Z$ in $B_{p,\alpha}^o$.

Proof. From (ii) we have that each subsequence of $(W_{f,n})$ has further subsequence that converges in distribution. If $W_{f,n'} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Y'$ and $W_{f,n''} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Y''$ then we have that $\lambda_r(W_{f,n'}) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \lambda_r(Y')$ and $\lambda_r(W_{f,n''}) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \lambda_r(Y'')$ for any dyadic number r . But (i) gives that both $\lambda_r(Y')$ and $\lambda_r(Y'')$ have the same distribution as $\lambda_r(Z)$. Since Schauder coefficients $(\lambda_r(Z))$ determines the distribution of Z we can conclude that Y' and Y'' are equally distributed with Z . This ends the proof. \square

For polygonal line processes build from any stationary sequence the tightness conditions given in Theorem 2.5 can be simplified.

Theorem 3.2. *Let $p \geq 1$ and $\alpha > 1/p$. The sequence $(W_{f,n})$ is tight in $B_{p,\alpha}^o$ provided that*

$$\lim_{J \rightarrow +\infty} \limsup_{N \rightarrow +\infty} \sum_{j=J}^N 2^j \int_0^1 x^{p-1} \mathbb{P}\left(2^{-(N-j)/2} \max_{1 \leq k \leq 2^{N-j}} |S_{f,k}| > x 2^{j/q(p,\alpha)}\right) dx = 0. \quad (3.1.1)$$

Proof. Assume that f satisfies (3.1.1). We have to show that $(W_{f,n})$ satisfies the conditions of Theorem 2.5. First we check its condition (i). Since

$$\|W_{f,n}\|_p \leq \sup_{0 \leq t \leq 1} |W_{f,n}(t)| = n^{-1/2} \max_{0 \leq t \leq 1} |S_{f,k}|,$$

the proof of (i) reduces to

$$\lim_{b \rightarrow +\infty} \sup_{N \geq 1} \mathbb{P}\left(2^{-N/2} \max_{1 \leq k \leq 2^N} |S_{f,k}| > b\right) = 0. \quad (3.1.2)$$

Notice that (3.1.1) implies (by considering the term of index J in the sum) that

$$\lim_{J \rightarrow +\infty} \limsup_{N \rightarrow +\infty} 2^J \mathbb{P}\left(2^{-(N-J)/2} \max_{1 \leq k \leq 2^{N-J}} |S_{f,k}| > \frac{1}{2} 2^{J/q(p,\alpha)}\right) = 0,$$

and consequently

$$\lim_{J \rightarrow +\infty} \limsup_{N \rightarrow +\infty} 2^J \mathbf{P} \left(2^{-N/2} \max_{1 \leq k \leq 2^N} |S_{f,k}| > 2^{J/q(p,\alpha)} \right) = 0.$$

For a fixed ε , we choose J_0 such that

$$\limsup_{N \rightarrow +\infty} 2^{J_0} \mathbf{P} \left(2^{-N/2} \max_{1 \leq k \leq 2^N} |S_{f,k}| > 2^{J_0/q(p,\alpha)} \right) < 2\varepsilon.$$

There exists an integer N_0 such that for $N \geq N_0$,

$$\mathbf{P} \left(2^{-N/2} \max_{1 \leq k \leq 2^N} |S_{f,k}| > 2^{J_0/q(p,\alpha)} \right) < \varepsilon.$$

Since $\sup_{N \leq N_0} \mathbf{P} \left(2^{-N/2} \max_{1 \leq k \leq 2^N} |S_{f,k}| > b \right) \rightarrow 0$ as b goes to infinity, we can choose b'_0 such that $\max_{N \leq N_0} \mathbf{P} \left(2^{-N/2} \max_{1 \leq k \leq 2^N} |S_{f,k}| > b'_0 \right) < \varepsilon$. Taking $b_0 := \max(2^{J_0/q(p,\alpha)}/2, b'_0)$, we have for $b \geq b_0$,

$$\sup_{N \geq 1} \mathbf{P} \left(2^{-N/2} \max_{1 \leq k \leq 2^N} |S_{f,k}| > b \right) < \varepsilon,$$

which proves (3.1.2) and the same time (i) of Theorem 2.5.

Now, let us prove condition (ii) of Theorem 2.5. Since

$$\begin{aligned} & \sum_{j=J}^N 2^j \int_0^1 x^{p-1} \mathbf{P} \left(2^{-(N-j)/2} \max_{1 \leq k \leq 2^{N-j}} |S_{f,k}| > x 2^{j/q(p,\alpha)} \right) dx \\ & \geq 2^N \int_{1/2}^{3/4} x^{p-1} \mathbf{P} \left(|f| > x 2^{N/q(p,\alpha)} \right) dx \\ & \geq 2^N (1/2)^{p-1} \mathbf{P} \left(|f| > \frac{3}{4} 2^{N/q(p,\alpha)} \right), \end{aligned}$$

we infer that condition (3.1.1) implies

$$\lim_{t \rightarrow +\infty} t^{q(p,\alpha)} \mathbf{P} \left(|f| > t \right) = 0. \quad (3.1.3)$$

We first prove that for each positive ε ,

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{j \geq [\log n] + 1} 2^{j\alpha-j/p} \left(\sum_{r \in D_j} |\lambda_r(W_{f,n})|^p \right)^{1/p} > \varepsilon \right) = 0.$$

We shall actually prove that

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{j \geq [\log n] + 1} 2^{j\alpha-j/p} \left(\sum_{r \in D_j} |W_{f,n}(r^+) - W_{f,n}(r)|^p \right)^{1/p} > \varepsilon \right) = 0, \quad (3.1.4)$$

since the differences $W_{f,n}(r) - W_{f,n}(r^-)$ can be treated similarly. To this aim, define for a fixed $j \geq [\log n] + 1$ the sets

$$I_k := \left(r \in D_j, \frac{k}{n} \leq r < r^+ < \frac{k+1}{n} \right), \quad 0 \leq k \leq n-1;$$

and

$$J_k := \left(r \in D_j, \frac{k}{n} \leq r < \frac{k+1}{n} \leq r^+ < \frac{k+2}{n} \right), \quad 0 \leq k \leq n-2.$$

Assume that r belongs to I_k . Then $[nr] = [nr^+] = k$. We thus have

$$|W_{f,n}(r^+) - W_{f,n}(r)| = |(nr^+ - nr) f \circ \mathbf{T}^k / \sqrt{n}| = n^{1/2} 2^{-j} |f \circ \mathbf{T}^k|. \quad (3.1.5)$$

Now, assume that r belongs to J_k . Then

$$\begin{aligned} |W_{f,n}(r^+) - W_{f,n}(r)| &\leq |W_{f,n}(r^+) - W_{f,n}((k+1)/n)| + |W_{f,n}((k+1)/n) - W_{f,n}(r)| \\ &= n^{-1/2} (|(nr^+ - (k+1)) f \circ \mathsf{T}^{k+1}| + |f \circ \mathsf{T}^k - (nr - k) f \circ \mathsf{T}^k|), \end{aligned}$$

and using the fact that $0 \leq nr^+ - (k+1) \leq nr^+ - nr = n2^j$ and $0 \leq 1 - (nr - k) \leq (nr^+ - k) - (nr - k) = n2^{-j}$, we get

$$|W_{f,n}(r^+) - W_{f,n}(r)| \leq \sqrt{n}2^{-j} (|f \circ \mathsf{T}^k| + |f \circ \mathsf{T}^{k+1}|). \quad (3.1.6)$$

Since $D_j = \{1 - 2^{-j}\} \cup \bigcup_{k=0}^{n-1} I_k \cup \bigcup_{k=0}^{n-2} J_k$ and for $r = 1 - 2^{-j}$,

$$|W_{f,n}(r^+) - W_{f,n}(r)| \leq n^{-1/2}2^{-j} |f \circ \mathsf{T}^n|,$$

we have in view of (3.1.5) and (3.1.6),

$$\begin{aligned} \left(\sum_{r \in D_j} |W_{f,n}(r^+) - W_{f,n}(r)|^p \right)^{1/p} &\leq n^{-1/2}2^{-j} |f \circ \mathsf{T}^n| + n^{1/2}2^{-j} \left(\sum_{k=0}^{n-1} \text{Card}(I_k) |f \circ \mathsf{T}^k|^p \right)^{1/p} \\ &\quad + n^{1/2}2^{-j} \left(\sum_{k=0}^{n-2} \text{Card}(J_k) (|f \circ \mathsf{T}^k| + |f \circ \mathsf{T}^{k+1}|)^p \right)^{1/p}. \end{aligned}$$

We now have to bound $\text{Card}(I_k)$ and $\text{Card}(J_k)$. Let $1 \leq l \leq 2^j$. If $(2l-1)2^{-j}$ belongs to I_k , then we should have $2^j k/n \leq 2l-1 < 2l < 2^j(k+1)/n$ hence $2^{j-1}k/n \leq l < 2^{j-1}(k+1)/n$ and it follows that I_k cannot have more than $2^j/n$ elements. If $(2l-1)2^{-j}$ belongs to J_k , then we should have $2^j(k+1)/n \leq 2l < 2^j(k+2)/n$ and we deduce that the cardinal of J_k does not exceed $2^j/n$. Therefore, we have

$$\left(\sum_{r \in D_j} |W_{f,n}(r^+) - W_{f,n}(r)|^p \right)^{1/p} \leq 3n^{1/2}2^{-j} (2^j/n)^{1/p} \left(\sum_{k=0}^n |f \circ \mathsf{T}^k|^p \right)^{1/p}.$$

and

$$\begin{aligned} \sup_{j \geq \lfloor \log n \rfloor + 1} 2^{j\alpha-j/p} \left(\sum_{r \in D_j} |\lambda_r(W_{f,n})|^p \right)^{1/p} &\leq 3 \sup_{j \geq \lfloor \log n \rfloor + 1} n^{1/2}2^{-j}2^{j\alpha-j/p} \left(\sum_{k=0}^n |f \circ \mathsf{T}^k|^p \right)^{1/p} \\ &\leq n^{-1/2+\alpha-1/p} \left(\sum_{k=0}^n |f \circ \mathsf{T}^k|^p \right)^{1/p} = n^{-1/q(p,\alpha)} \left(\sum_{k=0}^n |f \circ \mathsf{T}^k|^p \right)^{1/p}. \end{aligned}$$

We thus have to prove that the latter term goes to zero in probability as n goes to infinity.

Lemma 3.3. *Let f be a function such that (3.1.3) holds. Then*

$$n^{-1/q(p,\alpha)} \left(\sum_{k=0}^n |f \circ \mathsf{T}^k|^p \right)^{1/p} \xrightarrow[n \rightarrow \infty]{\mathsf{P}} 0.$$

Proof. For fixed δ and n , define $f' := f\mathbf{1}(|f| \leq \delta n^{1/q(p,\alpha)})$ and $f'' = f - f'$. By Markov's inequality, we have with $q = q(p, \alpha)$

$$\begin{aligned} \mathsf{P} \left(n^{-1/q} \left(\sum_{k=0}^n |f' \circ \mathsf{T}^k|^p \right)^{1/p} > \varepsilon \right) &\leq \varepsilon^{-p} n^{-p/q} \sum_{k=0}^n \mathsf{E} |f' \circ \mathsf{T}^k|^p \\ &\leq 2\varepsilon^{-p} n^{1-p/q} \mathsf{E} |f'|^p. \end{aligned} \quad (3.1.7)$$

Now, note that

$$\begin{aligned} \mathbb{E} |f'|^p &= p \int_0^{\delta n^{1/q}} t^{p-1} \mathbb{P}(|f'| > t) dt \leq p \int_0^{\delta n^{1/q}} t^{p-q-1} dt \cdot \sup_{s>0} s^q \mathbb{P}(|f| > s) \\ &= \frac{p}{p-q} \delta^{(p-q)/q} n^{(p-q)/q} \cdot \sup_{s>0} s^q \mathbb{P}(|f| > s), \end{aligned}$$

hence by (3.1.7), we have

$$\mathbb{P}\left(n^{-1/q} \left(\sum_{k=0}^n |f' \circ \mathsf{T}^k|^p \right)^{1/p} > \varepsilon\right) \leq \varepsilon^{-p} \frac{2p}{p-q} \delta^{(p-q)/q}. \quad (3.1.8)$$

Notice also that

$$\mathbb{P}\left(n^{-1/q} \left(\sum_{k=0}^n |f'' \circ \mathsf{T}^k|^p \right)^{1/p} > \varepsilon\right) \leq (n+1) \mathbb{P}(|f| > \delta n^{1/q}). \quad (3.1.9)$$

The combination of (3.1.8) and (3.1.9) gives

$$\limsup_{n \rightarrow +\infty} \mathbb{P}\left(n^{-1/q} \left(\sum_{k=0}^n |f \circ \mathsf{T}^k|^p \right)^{1/p} > \varepsilon\right) \leq \varepsilon^{-p} \frac{2p}{p-q} \delta^{(p-q)/q}$$

and since δ is arbitrary and $p > q$, this concludes the proof of Lemma 3.3. \square

An application of the Lemma 3.3 gives (3.1.4). Now, we have to prove that

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\sup_{J \leq j \leq \lfloor \log n \rfloor} 2^{j\alpha-j/p} \left(\sum_{r \in D_j} |W_{f,n}(r^+) - W_{f,n}(r)|^p \right)^{1/p} > \varepsilon\right) = 0.$$

It suffices to prove that

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(n^{-1/2} \sup_{J \leq j \leq \lfloor \log n \rfloor} 2^{j\alpha-j/p} \left(\sum_{r \in D_j} |S_{\lfloor nr^+ \rfloor} - S_{\lfloor nr \rfloor}|^p \right)^{1/p} > \varepsilon\right) = 0. \quad (3.1.10)$$

Indeed, we have

$$|W_{f,n}(r^+) - W_{f,n}(r)| \leq n^{-1/2} \left(|S_{f, \lfloor nr^+ \rfloor} - S_{f, \lfloor nr \rfloor}| + |f \circ \mathsf{T}^{\lfloor nr^+ \rfloor}| + |f \circ \mathsf{T}^{\lfloor nr \rfloor}| \right),$$

and for $j \leq \lfloor \log n \rfloor$, $2^j \leq n$, so that the set $\{\lfloor nr^+ \rfloor, \lfloor nr \rfloor, r \in D_j\}$ consists of distinct elements. Therefore,

$$\sup_{1 \leq j \leq \lfloor \log n \rfloor} 2^{j\alpha-j/p} \left(\sum_{r \in D_j} \left[n^{-1/2} |f \circ \mathsf{T}^{\lfloor nr^+ \rfloor}| + |f \circ \mathsf{T}^{\lfloor nr \rfloor}| \right]^p \right)^{1/p} \leq 2n^{-1/q} \left(\sum_{k=0}^n |f \circ \mathsf{T}^k|^p \right)^{1/p},$$

and this quantity goes to zero in probability by Lemma 3.3. The proof of (3.1.10) reduces to establish that for each positive ε ,

$$\lim_{J \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{j=J}^{\lfloor \log n \rfloor} \mathbb{P}(A_{n,j}) = 0, \quad (3.1.11)$$

where

$$A_{n,j} := \left\{ \sum_{l=1}^{2^{j-1}} |S_{\lfloor n2^{l-1} \rfloor}(f) - S_{\lfloor n(2^{l-1}-1) \rfloor}(f)|^p > \varepsilon n^{p/2} 2^{j(1-p\alpha)} \right\}.$$

We now bound $\mathbf{P}(A_{n,j})$ by splitting the probability over the set

$$B_{n,j} := \bigcup_{l=1}^{2^{j-1}} \left\{ |S_{f, \lfloor n2l2^{-j} \rfloor} - S_{f, \lfloor n(2l-1)2^{-j} \rfloor}| > n^{1/2} 2^{j(1/p-\alpha)} \right\}.$$

One bounds $\mathbf{P}(A_{n,j} \cap B_{n,j})$ by $\mathbf{P}(B_{n,j})$, which can in turn be bounded by

$$\sum_{l=1}^{2^{j-1}} \mathbf{P} \left(|S_{f, \lfloor n2l2^{-j} \rfloor} - S_{f, \lfloor n(2l-1)2^{-j} \rfloor}| > n^{1/2} 2^{j(1/p-\alpha)} \right)$$

and thanks to stationarity and the fact that

$$\lfloor n2l2^{-j} \rfloor - \lfloor n(2l-1)2^{-j} \rfloor \leq n2^{-j} + 1 \leq 2n2^{-j}, \quad (3.1.12)$$

we obtain

$$\begin{aligned} \mathbf{P}(A_{n,j} \cap B_{n,j}) &\leq 2^{j-1} \mathbf{P} \left(\max_{1 \leq k \leq \lfloor 2n2^{-j} \rfloor} |S_{f,k}| > n^{1/2} 2^{j(1/p-\alpha)} \right) \\ &\leq 2^{j-1} 2^{p-1} \int_{1/2}^1 t^{p-1} \mathbf{P} \left(\max_{1 \leq k \leq \lfloor 2n2^{-j} \rfloor} |S_{f,k}| > tn^{1/2} 2^{j(1/p-\alpha)} \right) dt. \end{aligned} \quad (3.1.13)$$

Now, in order to bound $\mathbf{P}(A_{n,j} \cap B_{n,j}^c)$, we start by the pointwise inequalities

$$\begin{aligned} \varepsilon n^{p/2} 2^{j(1-p\alpha)} \mathbf{1}(A_{n,j} \cap B_{n,j}^c) &\leq \sum_{l=1}^{2^{j-1}} |S_{f, \lfloor n2l2^{-j} \rfloor} - S_{f, \lfloor n(2l-1)2^{-j} \rfloor}|^p \mathbf{1}(A_{n,j} \cap B_{n,j}^c) \\ &\leq \sum_{l=1}^{2^{j-1}} |S_{f, \lfloor n2l2^{-j} \rfloor} - S_{f, \lfloor n(2l-1)2^{-j} \rfloor}|^p \mathbf{1} \left\{ |S_{f, \lfloor n2l2^{-j} \rfloor} - S_{f, \lfloor n(2l-1)2^{-j} \rfloor}| \leq n^{1/2} 2^{j(1/p-\alpha)} \right\}. \end{aligned}$$

Integrating and using the fact that for a non-negative random variable Y and a positive R ,

$$\mathbf{E}(Y^p \mathbf{1}\{Y \leq R\}) = pR^p \int_0^1 t^{p-1} \mathbf{P}(Y > Rt) dt,$$

we derive by stationarity and (3.1.12) that

$$\mathbf{P}(A_{n,j} \cap B_{n,j}^c) \leq \frac{p}{\varepsilon} 2^{j-1} \int_0^1 t^{p-1} \mathbf{P} \left(\max_{1 \leq k \leq \lfloor 2n2^{-j} \rfloor} |S_{f,k}| > tn^{1/2} 2^{j(1/p-\alpha)} \right) dt. \quad (3.1.14)$$

Let us denote by K a constant depending only on p and ε which may change from line to line.

By (3.1.13) and (3.1.14), we derive that

$$\sum_{j=J}^{\lfloor \log n \rfloor} \mathbf{P}(A_{n,j}) \leq K \sum_{j=J}^{\lfloor \log n \rfloor} 2^j \int_0^1 t^{p-1} \mathbf{P} \left(\max_{1 \leq k \leq \lfloor n2^{1-j} \rfloor} |S_{f,k}| > tn^{1/2} 2^{j(1/p-\alpha)} \right) dt.$$

If $2^N \leq n < 2^{N+1}$, then we have

$$\sum_{j=J}^{\lfloor \log n \rfloor} \mathbf{P}(A_{n,j}) \leq K \sum_{j=J}^N 2^j \int_0^1 t^{p-1} \mathbf{P} \left(\max_{1 \leq k \leq 2^{N+2-j}} |S_{f,k}| > t2^{N/2} 2^{j(1/p-\alpha)} \right) dt,$$

hence

$$\sum_{j=J}^{\lfloor \log n \rfloor} \mathbf{P}(A_{n,j}) \leq K \sum_{j=J}^{N+2} 2^j \int_0^1 s^{p-1} \mathbf{P} \left(\max_{1 \leq k \leq 2^{N+2-j}} |S_{f,k}| > s2^{\frac{N+2}{2}} 2^{j(1/p-\alpha)} \right) ds.$$

Splitting the integral into two parts, we infer that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \sum_{j=J}^{\lfloor \log n \rfloor} \mathbb{P}(A_{n,j}) \\ \leq K \limsup_{N \rightarrow +\infty} \sum_{j=J}^{N+2} 2^j \int_0^1 s^{p-1} \mathbb{P} \left(\max_{1 \leq k \leq 2^{N+2-j}} |S_{f,k}| > s 2^{(N+2)/2} 2^{j(1/p-\alpha)} \right) ds \end{aligned} \quad (3.1.15)$$

and the limit of the latter quantity as J goes to infinity is zero by (3.1.1). This concludes the proof of Theorem 3.2. \square

Remark 3.4. Using deviation inequalities, similar results as those found for the Hölderian weak invariance principle for stationary mixing and τ -dependent sequences in [4] can be found for Besov spaces.

Lemma 3.5 (Proposition 3.5 in [5]). *For any $q > 2$, there exists a constant $c(q)$ such that if $(f \circ \mathbb{T}^i)_{i \geq 0}$ is a martingale differences sequence with respect to the filtration $(\mathbb{T}^{-i} \mathcal{F}_0)_{i \geq 0}$ then for each integer $n \geq 1$,*

$$\begin{aligned} \mathbb{P} \left(\frac{1}{\sqrt{n}} \max_{1 \leq i \leq n} |S_{f,i}| \geq t \right) \leq c(q) n \int_0^1 \mathbb{P} \left(|f| \geq \sqrt{n} u t \right) u^{q-1} du + \\ + c(q) \int_0^\infty \mathbb{P} \left((\mathbb{E}(f^2 | \mathbb{T} \mathcal{F}_0))^{1/2} > v t \right) \min(v, v^{q-1}) dv. \end{aligned} \quad (3.1.16)$$

3.2. Proof of Theorem 1.1. According to Lemma 3.1 we need only to prove that the sequence $(n^{-1/2} \zeta_{f,n})_{n \geq 1}$ is tight in $B_{p,\alpha}^o$. To this aim, we have to check the condition (3.1.1) of Theorem 3.2. For fixed N and J such that $N \geq J$, $j \in (J, \dots, N)$ and $x \in [0, 1]$, we have by (3.1.16) of Lemma 3.5,

$$\begin{aligned} \mathbb{P} \left(2^{-\frac{N-j}{2}} \max_{1 \leq k \leq 2^{N-j}} |S_{f,k}| > x 2^{\frac{j}{q(p,\alpha)}} \right) \\ \leq c(q) 2^{N-j} \int_0^1 \mathbb{P} \left(|f| \geq 2^{\frac{N-j}{2}} x 2^{\frac{j}{q(p,\alpha)}} u \right) u^{q-1} du \\ + c(q) \int_0^\infty \mathbb{P} \left((\mathbb{E}(f^2 | \mathbb{T} \mathcal{F}_0))^{1/2} > v x 2^{\frac{j}{q(p,\alpha)}} \right) \min(v, v^{q-1}) dv, \end{aligned} \quad (3.2.1)$$

from which we infer that

$$\begin{aligned} \sum_{j=J}^N 2^j \int_0^1 x^{p-1} \mathbb{P} \left(2^{-\frac{N-j}{2}} \max_{1 \leq k \leq 2^{N-j}} |S_{f,k}| > x 2^{\frac{j}{q(p,\alpha)}} \right) dx \\ \leq c(q) 2^N \sum_{j=J}^N \int_0^1 x^{p-1} \int_0^1 \mathbb{P} \left(|f| \geq 2^{\frac{N-j}{2}} x 2^{j(1/p-\alpha)} u \right) dx u^{q-1} du \\ + c(q) \int_0^\infty \int_0^1 x^{p-1} \sum_{j=J}^N 2^j \mathbb{P} \left((\mathbb{E}(f^2 | \mathbb{T} \mathcal{F}_0))^{1/2} > v x 2^{\frac{j}{q(p,\alpha)}} \right) \min(v, v^{q-1}) dv dx \\ =: A(N, J) + B(N, J). \end{aligned} \quad (3.2.2)$$

Using the fact that

$$\mathbf{P}\left(|f| \geq t\right) \leq t^{-q(p,\alpha)} \sup_{s \geq t} s^{q(p,\alpha)} \mathbf{P}\left(|f| \geq s\right),$$

we derive the bound

$$A(N, J) \leq c(q) 2^{N(1-q(p,\alpha)/2)} \sum_{j=0}^N 2^{j(1/p-\alpha)} \int_0^1 \int_0^1 x^{p-q(p,\alpha)-1} u^{q-q(p,\alpha)-1} \sup_{s \geq 2^{\frac{N}{2}} x u 2^{j(1/p-\alpha)}} \left(s^{q(p,\alpha)} \mathbf{P}\left(|f| \geq s\right) \right) dx du.$$

Since $j \leq N$, we have $2^{\frac{N}{2}} x u 2^{j(1/p-\alpha)} \geq x u 2^{N/q(p,\alpha)}$ and accounting the inequality $\sum_{j=0}^N 2^{j(1/p-\alpha)} \leq 2^{N(1/p-\alpha)}/(1-2^{1/p-\alpha})$, we obtain

$$A(N, J) \leq c(q) \int_0^1 \int_0^1 x^{p-q(p,\alpha)-1} u^{q-q(p,\alpha)-1} \sup_{s \geq 2^{\frac{N}{q(p,\alpha)}} x u} \left(s^{q(p,\alpha)} \mathbf{P}\left(|f| \geq s\right) \right) dx du.$$

Since $p > q(p, \alpha)$ and $q > q(p, \alpha)$, the integral $\int_0^1 \int_0^1 u^{q-q(p,\alpha)-1} x^{p-q(p,\alpha)-1} dx du$ is convergent and we infer by the monotone convergence theorem that

$$\forall J \geq 1, \quad \lim_{N \rightarrow +\infty} A(N, J) = 0. \quad (3.2.3)$$

Now, in order to control $B(N, J)$, we use the following elementary inequality: if Y is a non-negative random variable, then for each $J \geq 1$,

$$\sum_{j \geq J} 2^j \mathbf{P}\left(Y \geq 2^{j/q(p,\alpha)}\right) \leq 2 \mathbf{E}\left(Y^{q(p,\alpha)} \mathbf{1}\left(Y \geq 2^{J/q(p,\alpha)}\right)\right).$$

Applying this to $Y := (\mathbf{E}(f^2 | \mathcal{T}\mathcal{F}_0))^{1/2}/(vx)$, we obtain that

$$B(N, J) \leq c(q) \int_0^\infty \int_0^1 x^{p-q(\alpha)-1} \mathbf{E}\left([\mathbf{E}(f^2 | \mathcal{T}\mathcal{F}_0)]^{q/2} \mathbf{1}\left((\mathbf{E}(f^2 | \mathcal{T}\mathcal{F}_0))^{1/2} \geq vx 2^{J/q(p,\alpha)}\right)\right) \cdot \min(v, v^{q-1}) v^{-q(p,\alpha)} dv dx.$$

Here again, we conclude by monotone convergence that

$$\lim_{J \rightarrow +\infty} \sup_{N \geq 1} B(N, J) = 0, \quad (3.2.4)$$

since the integrals $\int_0^1 x^{p-q(\alpha)-1} dx$ and $\int_0^{+\infty} \min(v, v^{q-1}) v^{-q(p,\alpha)} dv$ are finite (as $q > q(\alpha)$).

Tightness of the sequence $(W_{f,n})_{n \geq 1}$ now follows from Theorem 3.2 and the combination of (3.2.2), (3.2.3) and (3.2.4). Accounting Lemma 3.1 this concludes the proof of Theorem 1.1.

3.3. Proof of Theorem 1.2. Sufficiency of the condition is contained in Theorem 1.1. Indeed, we represent the sequence $(Y_j)_{j \geq 0}$ by $(f \circ \mathcal{T}^j)_{j \in \mathbb{Z}}$, that is, $(f \circ \mathcal{T}^j)_{j \in \mathbb{Z}}$ is an i.i.d. sequence and Y_j has the same distribution as $f \circ \mathcal{T}^j$. To this aim we define $\Omega = \mathbb{R}^{\mathbb{Z}}$, $\mathcal{F} = \mathcal{B}^{\mathbb{Z}}$ and $\mathbf{P} = \mathbf{P}_Y^{\mathbb{Z}}$, where \mathbf{P}_Y is the distribution of Y_0 . Let $f((\omega_j)) = \omega_0$ for $(\omega_j) \in \mathbb{R}^{\mathbb{Z}}$ and let $\mathcal{T} : \Omega \rightarrow \Omega$ be the shift operator: $\mathcal{T}((\omega_j)) = (\omega_{j+1})$. Next let $\mathcal{F}_0 := \sigma(f \circ \mathcal{T}^j, j \leq 0)$. Then $\mathcal{T}\mathcal{F}_0 \subset \mathcal{F}_0$ and $\mathbf{E}(f | \mathcal{T}\mathcal{F}_0) = 0$, since f is independent of $\mathcal{T}\mathcal{F}_0$ and centered. Moreover, $\mathbf{E}(f^2 | \mathcal{T}\mathcal{F}_0) = \mathbf{E}(f^2)$, again by independence. Therefore, condition (ii) of (1.1) is satisfied. Since \mathcal{I} is trivial, $\mathbf{E}(f^2 | \mathcal{I}) = \mathbf{E}(f^2)$, which gives the convergence (1.0.2).

Let us prove the necessity of (1.0.3) for the invariance principle in $B_{p,\alpha}^o$. Since the space $B_{p,\alpha}^o$ is a separable Banach space, the sequence $(W_{2^n})_{n \geq 1}$ is tight in $B_{p,\alpha}^o$. Using Theorem 1 in [13], we can find for any positive η a number J_0 such that

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{j \geq J_0} 2^{j\alpha-j/p} \left(\sum_{r \in D_j} |W_{2^n}(r^+) - W_{2^n}(r)|^p \right)^{1/p} > \varepsilon \right) \leq \eta.$$

Therefore, if n is large enough, we have

$$\mathbb{P} \left(2^{n\alpha-n/p} \left(\sum_{r \in D_n} |W_{2^n}(r^+) - W_{2^n}(r)|^p \right)^{1/p} > \varepsilon \right) \leq 2\eta.$$

Since

$$\sum_{r \in D_n} |W_{2^n}(r^+) - W_{2^n}(r)|^p = 2^{-np/2} \sum_{l=1}^{2^{n-1}} |S_{2l} - S_{2l-1}|^p = 2^{-np/2} \sum_{l=1}^{2^{n-1}} |X_{2l-1}|^p,$$

we have the convergence in probability of the sequence $\left(2^{n\alpha-j/p} 2^{-np/2} \sum_{l=1}^{2^{n-1}} |X_{2l-1}|^p \right)$ to 0. By [7], this implies that $2^{n\alpha-j/p} 2^{-np/2} \mathbb{P}(|X_1| > 2^n) \rightarrow 0$, hence (1.0.3) holds. This ends the proof of Theorem 1.2.

3.4. Proof of Theorem 1.3. We first start by a lemma which guarantees the lack of tightness of the partial sum process.

Lemma 3.6. *Let $1/p < \alpha < 1/2$ and let f be a function such that there exist increasing sequences of real numbers $(n_l)_{l \geq 1}$ and $(k_l)_{l \geq 1}$ satisfying the following properties: $k_l/n_l \rightarrow 0$ as l goes to infinity and*

$$\inf_{l \geq 1} \mathbb{P} \left(\frac{1}{n_l^{q(p,\alpha)}} \max_{1 \leq k \leq k_l} \frac{1}{k^\alpha} \left(\sum_{i=0}^{n_l-k} |S_{f,i+k} - S_{f,i}|^p \right)^{1/p} > 1 \right) > 0,$$

where $q(p,\alpha)$ is given by (1.0.1). Then the sequence $(W_n(f))_{n \geq 1}$ is not tight in $B_{p,\alpha}^o$.

Proof. If the sequence $(W_n(f))_{n \geq 1}$ was tight in $B_{p,\alpha}^o$, then we would be able to extract a weakly convergence subsequence of $(W_{n_l}(f))_{l \geq 1}$. Therefore, we can assume without loss of generality that $(W_{n_l}(f))_{l \geq 1}$ converges in distribution in $B_{p,\alpha}$. Consequently, the sequence $\left(\sup_{|t| \leq k_l/n_l} t^{-\alpha} \omega_p(W_{n_l}, t) \right)_{l \geq 1}$ should converge to 0 in probability as l goes to infinity. But

$$\sup_{|t| \leq k_l/n_l} t^{-\alpha} \omega_p(W_{n_l}, t) \geq \frac{c(p)}{n_l^{q(p,\alpha)}} \max_{1 \leq k \leq k_l} \frac{1}{k^\alpha} \left(\sum_{i=0}^{n_l-k} |S_{f,i+k} - S_{f,i}|^p \right)^{1/p}$$

for some constant depending only on p (this can be seen by restricting the supremum over the t of the form k/n_l where $1 \leq k \leq k_l$).

□

Let us recall the statement of Lemma 3.8 in [8].

Lemma 3.7. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T})$ be an ergodic probability measure preserving system of positive entropy. There exists two \mathbb{T} -invariant sub- σ -algebras \mathcal{B} and \mathcal{C} of \mathcal{F} and a function $g: \Omega \rightarrow \mathbb{R}$ such that:*

- the σ -algebras \mathcal{B} and \mathcal{C} are independent;

- the function g is \mathcal{B} -measurable, takes the values $-1, 0$ and 1 , has zero mean and the process $(g \circ \mathbb{T}^n)_{n \in \mathbb{Z}}$ is independent;
- the dynamical system $(\Omega, \mathcal{C}, \mathbb{P}, \mathbb{T})$ is aperiodic.

In the sequel, we shall assume for simplicity that $\mathbb{P}(g = 1) = \mathbb{P}(g = -1) = 1/2$.

The construction follows the lines of that of Theorem 2.1 in [6]. We define three increasing sequences of positive integers $(I_l)_{l \geq 1}$, $(J_l)_{l \geq 1}$, $(N_l)_{l \geq 1}$ and a sequence of real numbers $(L_l)_{l \geq 1}$ such that

$$\sum_{l=1}^{\infty} \frac{1}{L_l} < \infty \text{ and}$$

L_l is a continuity point of the cumulative distribution function of the random variable $2^{-1}Y_{1/2,1}$, which is defined in (5.0.1). Now, we define a sequence of real numbers $(J_l)_{l \geq 1}$ in such a way that for each $l \geq 1$,

$$\left| J_l \mathbb{P} \left(\left(Y_{1/2,1} > 2L_l \right) \right) - 7/8 \right| \leq 1/16. \quad (3.4.1)$$

Now, by Proposition 5.1, we can choose for each $l \geq 1$ an integer I_l such that

$$\forall n \geq I_l, \quad \left| \mathbb{P} \left(\left(Y_{n,1/2,1}(g) > 2L_l \right) \right) - \mathbb{P} \left(\left(Y_{1/2,1} > 2L_l \right) \right) \right| \leq \frac{1}{lJ_l}. \quad (3.4.2)$$

Let $K_l := 2^{I_l+J_l}$. We define the sequence $(N_l)_{l \geq 1}$ in such a way that for each $l \geq 1$,

$$\frac{1}{N_l^{1/q(p,\alpha)}} K_l^{1-\alpha} \sum_{u=1}^{l-1} \left(\frac{N_u}{2^{I_u}} \right)^{1/q(p,\alpha)} \leq 1 \text{ and} \quad (3.4.3)$$

$$N_l \sum_{u=l+1}^{+\infty} K_u/N_u < \frac{1}{16}. \quad (3.4.4)$$

Using Rokhlin's lemma, we can find for any integer $l \geq 1$ a measurable set $C_l \in \mathcal{C}$ such that the sets $\mathbb{T}^i C_l$, $i = 0, \dots, N_l - 1$ are pairwise disjoint and $\mathbb{P} \left(\bigcup_{i=0}^{N_l-1} \mathbb{T}^i C_l \right) > 1/2$. We define for $l \geq 1$

$$f_l := \frac{1}{L_l} \sum_{j=1}^{J_l} \left(\frac{N_l}{2^{I_l+j}} \right)^{1/q(p,\alpha)} \mathbf{1} \left(\bigcup_{i=2^{J_l+j}+1}^{2^{J_l+j+1}} \mathbb{T}^{N_l-i} C_l \right) \text{ and} \quad (3.4.5)$$

$$f := \sum_{l=1}^{+\infty} f_l, \quad m := g \cdot f. \quad (3.4.6)$$

Note that $\mathbb{P}(f_l \neq 0) \leq K_l/N_l$, hence by (3.4.4) and the Borel-Cantelli lemma, the function f is well defined almost everywhere. Define

$$\mathcal{F}_0 := \sigma(g \circ \mathbb{T}^i, i \leq 0) \vee \mathcal{C}.$$

Proposition 3.8. *The σ -algebra \mathcal{F}_0 satisfies $\mathbb{T}\mathcal{F}_0 \subset \mathcal{F}_0$. The function m defined by (3.4.5) and (3.4.6) is \mathcal{F}_0 -measurable and satisfies $\mathbb{E}[m \mid \mathbb{T}\mathcal{F}_0] = 0$ and $\lim_{t \rightarrow +\infty} t^{q(p,\alpha)} \mathbb{P}\{|m| > t\} = 0$.*

A proof can be found in [6]

It remains to prove that the sequence $(W_n(m))_{n \geq 1}$ is not tight in $B_{p,\alpha}^o$.

To this aim, we shall check the conditions of Lemma 3.6. We first show the following intermediate step.

Lemma 3.9. *For each integer $l \geq 1$,*

$$\mathbb{P}\left(\frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{g_{f_l, i+k}} - S_{g_{f_l, i}}|^p \right)^{1/p} > 2\right) > \frac{1}{8}.$$

Let $l \geq 1$ be fixed. Assume that ω belongs to $\mathbb{T}^{N_l-i_0}$ for some $i_0 \in \{K_l, \dots, N_l-1\}$. Let i be such that $i_0 - 2^{I_l+j+1} \leq i \leq i_0 - 2^{I_l+j} + 1$ for some $j \in \{1, \dots, J_l\}$. We have

$$f_l \circ \mathbb{T}^i(\omega) = \frac{1}{L_l} \left(\frac{N_l}{2^{I_l+j}} \right)^{1/q(p,\alpha)}.$$

Consequently, for any k such that $2^{I_l+j-1} < k \leq 2^{I_l+j}$ and each i such that $i_0 - 2^{I_l+j+1} \leq i \leq i_0 - k - 2^{I_l+j} + 1$, we have

$$|S_{g_{f_l, i+k}} - S_{g_{f_l, i}}| = \frac{1}{L_l} \left(\frac{N_l}{2^{I_l+j}} \right)^{1/q(p,\alpha)} |S_{g, i+k} - S_{g, i}|.$$

It thus follows that

$$\begin{aligned} & \frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{g_{f_l, i+k}} - S_{g_{f_l, i}}|^p \right)^{1/p} \mathbf{1}(\mathbb{T}^{N_l-i_0} C_l) \\ & \geq \max_{1 \leq j \leq J_l} \max_{2^{I_l+j-1} < k \leq 2^{I_l+j}} k^{-\alpha} \frac{1}{L_l} \left(\frac{1}{2^{I_l+j}} \right)^{1/q(p,\alpha)} \left(\sum_{i=i_0-2^{I_l+j+1}}^{i_0-k-2^{I_l+j}+1} |S_{g, i+k} - S_{g, i}|^p \right)^{1/p} \mathbf{1}(\mathbb{T}^{N_l-i_0} C_l), \end{aligned}$$

and using disjointness of the sets $\mathbb{T}^{N_l-i_0} C_l$, $K_l \leq i_0 \leq N_l-1$, we infer that

$$\begin{aligned} & \mathbb{P}\left(\frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{g_{f_l, i+k}} - S_{g_{f_l, i}}|^p \right)^{1/p} > 2\right) \\ & \geq \mathbb{P}\left(\frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{g_{f_l, i+k}} - S_{g_{f_l, i}}|^p \right)^{1/p} > 2 \cap \bigcup_{i_0=K_l}^{N_l-1} \mathbb{T}^{N_l-i_0} C_l\right) \\ & = \sum_{i_0=K_l}^{N_l-1} \mathbb{P}\left(\left\{ \frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{g_{f_l, i+k}} - S_{g_{f_l, i}}|^p \right)^{1/p} > 2 \right\} \cap \mathbb{T}^{N_l-i_0} C_l\right) \\ & \geq \sum_{i_0=K_l}^{N_l-1} \mathbb{P}(A_{i_0} \cap \mathbb{T}^{N_l-i_0} C_l) \quad (3.4.7) \end{aligned}$$

where

$$A_{i_0} = \left\{ \max_{1 \leq j \leq J_l} \max_{2^{I_l+j-1} < k \leq 2^{I_l+j}} k^{-\alpha} \frac{1}{L_l} \left(\frac{1}{2^{I_l+j}} \right)^{1/q(p,\alpha)} \left(\sum_{i=i_0-2^{I_l+j+1}}^{i_0-k-2^{I_l+j}+1} |S_{g, i+k} - S_{g, i}|^p \right)^{1/p} > 2 \right\}.$$

Since \mathbb{T} is measure-preserving, the events $A_{i_0} \cap \mathbb{T}^{N_l-i_0} C_l$, $K_l \leq i_0 \leq N_l-1$ have the same probability, which is equal to $\mathbb{P}(A_{K_l} \cap \mathbb{T}^{N_l-K_l} C_l)$. The events A_{K_l} and $\mathbb{T}^{N_l-K_l} C_l$ belong

respectively to \mathcal{B} and \mathcal{C} , hence they are independent. In view of (3.4.7), we obtain

$$\begin{aligned} \mathbb{P}\left(\frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{g_{f_l}, i+k} - S_{g_{f_l}, i}|^p \right)^{1/p} > 2\right) &\geq (N_l - K_l) \mathbb{P}(A_{K_l}) \mathbb{P}(C_l) \\ &\geq \mathbb{P}(A_{K_l})/2. \end{aligned} \quad (3.4.8)$$

Now, in order to control the latter term, we shall use the following lemma:

Lemma 3.10. *Let $(H_l)_{l \geq 1}$ be an increasing sequence of integers. Assume that for each $l \geq 1$, the family of events $(A_{l,j})_{1 \leq j \leq H_l}$ is independent and that $\sum_{j=1}^{H_l} \mathbb{P}(A_{l,j}) \in [3/4, 1]$. Then for each $l \geq 1$,*

$$\mathbb{P}\left(\bigcup_{j=1}^{H_l} A_{l,j}\right) \geq 1/4.$$

Proof of Lemma 3.10. By Bonferroni's inequality, we have for any $l \geq 1$,

$$\mathbb{P}\left(\bigcup_{j=1}^{H_l} A_{l,j}\right) \geq \sum_{j=1}^{H_l} \mathbb{P}(A_{l,j}) - \sum_{1 \leq i < j \leq H_l} \mathbb{P}(A_{l,i} \cap A_{l,j}).$$

Using independence of $(A_{l,j})_{1 \leq j \leq H_l}$, we derive that

$$\begin{aligned} \mathbb{P}\left(\bigcup_{j=1}^{H_l} A_{l,j}\right) &\geq \sum_{j=1}^{H_l} \mathbb{P}(A_{l,j}) - \frac{1}{2} \left(2 \sum_{1 \leq i < j \leq H_l} \mathbb{P}(A_{l,i}) \mathbb{P}(A_{l,j}) \right) \\ &= \sum_{j=1}^{H_l} \mathbb{P}(A_{l,j}) - \frac{1}{2} \left(\left(\sum_{j=1}^{H_l} \mathbb{P}(A_{l,j}) \right)^2 - \sum_{j=1}^{H_l} (\mathbb{P}(A_{l,j}))^2 \right) \\ &\geq \sum_{j=1}^{H_l} \mathbb{P}(A_{l,j}) - \frac{1}{2} \left(\sum_{j=1}^{H_l} \mathbb{P}(A_{l,j}) \right)^2 \\ &\geq 3/4 - 1/2 = 1/4. \end{aligned}$$

□

We now use Lemma 3.10 with the choices $H_l = J_l$ and

$$A_{l,j} := \left\{ \max_{2^{I_l+j-1} < k \leq 2^{I_l+j}} k^{-\alpha} \frac{1}{L_l} \left(\frac{1}{2^{I_l+j}} \right)^{1/q(p,\alpha)} \left(\sum_{i=K_l-2^{I_l+j+1}}^{K_l-k-2^{I_l+j+1}} |S_{g,i+k} - S_{g,i}|^p \right)^{1/p} > 2 \right\}.$$

We indeed have, with the notations of (5.0.1) and by (3.4.2),

$$\left| \sum_{j=1}^{J_l} \mathbb{P}(A_{l,j}) - J_l \mathbb{P}\left((Y_{1/2,1} > 2L_l)\right) \right| \leq \sum_{j=1}^{J_l} \frac{1}{lJ_l} = 1/l$$

hence by (3.4.1),

$$\left| \sum_{j=1}^{J_l} \mathbb{P}(A_{l,j}) - 7/8 \right| \leq \frac{1}{16l} + \left| J_l \mathbb{P}\left((Y_{1/2,1} > 2L_l)\right) - 7/8 \right| \leq \frac{1}{8}.$$

We get, in view of (3.4.8) the lower bound

$$\mathbb{P}\left(\frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{gf_l, i+k} - S_{gf_l, i}|^p \right)^{1/p} > 2\right) \geq \frac{1}{8}.$$

This concludes the proof of Lemma 3.9.

Now, we prove that for any $l \geq 1$,

$$\mathbb{P}\left(\frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{m, i+k} - S_{m, i}|^p \right)^{1/p} > 1\right) \geq \frac{1}{16}. \quad (3.4.9)$$

We first prove that

$$\frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{f'_l, i+k} - S_{f'_l, i}|^p \right)^{1/p} \leq 1, \quad (3.4.10)$$

where $f'_l := \sum_{i=1}^l gf_i$. First note that for $1 \leq k \leq K_l$ and $0 \leq i \leq N_l - k$,

$$\begin{aligned} |S_{f'_l, i+k} - S_{f'_l, i}| &\leq \sum_{u=1}^{l-1} |S_{gf_u, i+k} - S_{gf_u, i}| \leq k \sum_{u=1}^{l-1} \max_{i \leq v \leq i+k-1} |f_u \circ \mathbb{T}^v| \\ &\leq k \cdot \sum_{u=1}^{l-1} \max_{0 \leq v \leq N_l} |f_u \circ \mathbb{T}^v|, \end{aligned}$$

hence

$$\frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{f'_l, i+k} - S_{f'_l, i}|^p \right)^{1/p} \leq \frac{1}{N_l^{1/q(p,\alpha)}} K_l^{1-\alpha} \cdot \sum_{u=1}^{l-1} \max_{0 \leq v \leq N_l} |f_u \circ \mathbb{T}^v|.$$

Now, by definition of f_u , for each $\omega \in \Omega$, the following inequality holds: $|f_u(\omega)| \leq \left(\frac{N_u}{2^{I_u}}\right)^{1/q(p,\alpha)}$.

Consequently,

$$\frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{f'_l, i+k} - S_{f'_l, i}|^p \right)^{1/p} \leq \frac{1}{N_l^{1/q(p,\alpha)}} K_l^{1-\alpha} \sum_{u=1}^{l-1} \left(\frac{N_u}{2^{I_u}}\right)^{1/q(p,\alpha)},$$

and this term does not exceed 1 by (3.4.3). This proves (3.4.10)

Now, defining $f''_l := \sum_{u=l+1}^{+\infty} gf_u$, we have

$$\begin{aligned} &\mathbb{P}\left(\frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{f''_l, i+k} - S_{f''_l, i}|^p \right)^{1/p} \neq 0\right) \\ &\leq \sum_{u=l+1}^{+\infty} \mathbb{P}\left(\max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} |S_{gf_u, i+k} - S_{gf_u, i}|^p \right)^{1/p} \neq 0\right) \\ &\leq \sum_{u=l+1}^{+\infty} \mathbb{P}\left(\max_{0 \leq v \leq N_l-1} |gf_u \circ \mathbb{T}^v| \neq 0\right) \\ &\leq N_l \sum_{u=l+1}^{+\infty} \mathbb{P}\left(|gf_u| \neq 0\right) \leq N_l \sum_{u=l+1}^{+\infty} \mathbb{P}\left(f_u \neq 0\right). \end{aligned}$$

By constructing of f_u , we have $\mathbf{P}\left(\left(f_u \neq 0\right)\right) \leq K_u/N_u$, hence

$$\mathbf{P}\left(\frac{1}{N_l^{1/q(p,\alpha)}} \max_{1 \leq k \leq K_l} k^{-\alpha} \left(\sum_{i=0}^{N_l-k} \left|S_{f_l'', i+k} - S_{f_l'', i}\right|^p\right)^{1/p} \neq 0\right) \leq N_l \sum_{u=l+1}^{+\infty} \frac{K_u}{N_u} \leq 1/16, \quad (3.4.11)$$

by (3.4.4).

Thus (3.4.9) follows from the combination of Lemma 3.9, (3.4.10) and (3.4.11). This ends the proof of Theorem 1.3.

4. PROOFS: THE CASE $\alpha \leq 1/p$

We start with the following lemma which reduces the proof of convergence to that of tightness.

Lemma 4.1. *Let $p \geq 1$ and $0 \leq \alpha \leq \min\{1/2, 1/p\}$. Assume that Z is a random element in $B_{p,\alpha}^o$. Then for any stationary sequence $(f \circ \mathbf{T}^j)$ if*

- (i) $W_{f,n} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Z$ in $L_p[0, 1]$, and
- (ii) $(W_{f,n})$ is tight in $B_{p,\alpha}^o$,

then $W_{f,n} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Z$ in $B_{p,\alpha}^o$.

Proof. From (ii) we have that each subsequence of $(W_{f,n})$ has further subsequence that converges in distribution. If $W_{f,n'} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Y'$ and $W_{f,n''} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Y''$ then we have that for Franklin basis (f_k) it holds that $\langle W_{f,n'}, f_k \rangle \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \langle Y', f_k \rangle$ and $\langle W_{f,n''}, f_k \rangle \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \langle Y'', f_k \rangle$ for any k . But (i) gives that both $\langle Y', f_k \rangle$ and $\langle Y'', f_k \rangle$ have the same distribution as $\langle Z, f_k \rangle$. Since coefficients $\langle Z, f_k \rangle$ determines the distribution of Z we can conclude that Y' and Y'' are equally distributed with Z . This ends the proof. \square

4.1. Proof of Theorem 1.4. Due to continuity of the embedding $B_{2,\alpha}^o \hookrightarrow B_{p,\alpha}^o$ if $1 \leq p \leq 2$ and $0 \leq \alpha < 1/2$. it is enough to prove the case where either $p = 2$ and $0 \leq \alpha < 1/2$ or $p > 2$ and $0 \leq \alpha \leq 1/p$.

Recall $W_{f,n} = n^{-1/2} \zeta_{f,n}$. We shall prove for each $\varepsilon > 0$

$$\limsup_{\delta \rightarrow 0} \sup_{n \geq 1} I_n(\delta, \varepsilon) = 0, \quad (4.1.1)$$

where

$$I_n(\delta, \varepsilon) = \mathbf{P}\left(\delta^{-\alpha} \sup_{|h| \leq \delta} \left(\int_0^1 |W_{f,n}(t+h) - W_{f,n}(t)|^p dt\right)^{1/p} > \varepsilon\right).$$

Since the function $W_{f,n}(t)$, $0 \leq t \leq 1$ is affine in each interval $((k-1)/n, k/n]$, it holds for $s, t \in [(k-1)/n, k/n]$,

$$|W_{f,n}(s) - W_{f,n}(t)| \leq n^{1/2} |s - t| \cdot |f \circ \mathbf{T}^k|. \quad (4.1.2)$$

This observation leads to

$$\omega_p(W_{f,n}, \delta) \leq c_p [U_{f,n}(\delta) + V_{f,n}(\delta)],$$

where $c_p > 0$ is a constant depending on p only,

$$U_{f,n}(\delta) := \min\{\delta, n^{-1}\} n^{1/2} \left(\frac{1}{n} \sum_{k=1}^n |f \circ \mathbb{T}^k|^p \right)^{1/p},$$

$$V_{f,n}(\delta) := n^{-1/2} \max_{1 \leq \ell \leq \lfloor n\delta \rfloor} \left(\frac{1}{n} \sum_{k=0}^{n-\ell} \left| \sum_{j=k+1}^{k+\ell} f \circ \mathbb{T}^j \right|^p \right)^{1/p}.$$

As a consequence in order to establish (4.1.1) we have to prove

$$\lim_{\delta \rightarrow 0} \sup_{n \geq 1} \mathbf{P}(\delta^{-\alpha} U_{f,n}(\delta) \geq \varepsilon) = 0, \quad (4.1.3)$$

$$\lim_{\delta \rightarrow 0} \sup_{n \geq 1/\delta} \mathbf{P}(\delta^{-\alpha} V_{f,n}(\delta) > \varepsilon) = 0. \quad (4.1.4)$$

Consider first (4.1.3) and start with $p = 2$ and $0 \leq \alpha < 1/2$. By Chebyshev inequality

$$\begin{aligned} \mathbf{P}(\delta^{-\alpha} U_{f,n}(\delta) \geq \varepsilon) &\leq \varepsilon^{-2} \delta^{-2\alpha} \min\{\delta^2, n^{-2}\} \mathbf{E} \left(\sum_{k=1}^n |f \circ \mathbb{T}^k|^2 \right) \\ &\leq \varepsilon^{-2} \delta^{-2\alpha} \min\{\delta^2, n^{-2}\} n \mathbf{E}(f^2) \leq \varepsilon^{-2} \delta^{1-2\alpha} \end{aligned}$$

and (4.1.3) follows in this case. Now let $p > 2$ and $0 \leq \alpha \leq 1/p$. For this case we shall use truncation. Set for $\tau > 0$,

$$f' = f \mathbf{1}(|f| \leq \tau \sqrt{\max\{n, \delta^{-1}\}}), \quad f'' = f - f'.$$

Then $\mathbf{P}(\delta^{-\alpha} U_{f,n}(\delta) > \varepsilon) \leq n \mathbf{P}(|f| \geq \tau \sqrt{\max\{n, \delta^{-1}\}}) + \mathbf{P}(\delta^{-\alpha} U_{f',n}(\delta) > \varepsilon)$ and, since

$$n \mathbf{P}(|f| \geq \tau \sqrt{\max\{n, \delta^{-1}\}}) \leq \tau^{-2} \mathbf{E}(f)^2 \mathbf{1}(|f| \geq \sqrt{\delta^{-1}})$$

we reduce the proof of (4.1.3) to

$$\lim_{\delta \rightarrow 0} \sup_{n \geq 1} \mathbf{P}(\delta^{-\alpha} U_{f',n}(\delta) \geq \varepsilon) = 0. \quad (4.1.5)$$

We have by Chebyshev inequality,

$$\begin{aligned} \mathbf{P}(\delta^{-\alpha} U_{f',n}(\delta) \geq \varepsilon) &\leq \varepsilon^{-p} \delta^{-p\alpha} \min\{\delta^p, n^{-p}\} n^{p/2} \mathbf{E} \left[n^{-1} \sum_{k=1}^n |f' \circ \mathbb{T}^k|^p \right] \\ &\leq \varepsilon^{-p} \delta^{-p\alpha} \min\{\delta^p, n^{-p}\} n^{p/2} \mathbf{E}((f')^p) \\ &\leq \varepsilon^{-p} \delta^{-p\alpha} \min\{\delta^p, n^{-p}\} n^{p/2} \tau^{p-2} (\max\{n, \delta^{-1}\})^{(p-2)/2} \mathbf{E} f^2 \\ &\leq \varepsilon^{-p} \tau^{p-2} (\min\{\delta, n^{-1}\})^{1-p\alpha} \mathbf{E} f^2. \end{aligned}$$

Hence

$$\lim_{\delta \rightarrow 0} \sup_{n \geq 1} \mathbf{P}(\delta^{-\alpha} U_{f',n}(\delta) \geq \varepsilon) \leq \varepsilon^{-p} \tau^{p-2}.$$

Since $\tau > 0$ is arbitrary, the limit is indeed zero, and the proof of (4.1.5) is completed.

To prove (4.1.4) we start again with the case $p = 2$ and $0 \leq \alpha < 1/2$. In this case Chebyshev inequality along with stationarity and Doob-Kolmogorov inequality yields

$$\begin{aligned} \mathbb{P}(\delta^{-\alpha} V_{f,n}(\delta) > \varepsilon) &\leq \varepsilon^{-2} \delta^{-2\alpha} \mathbb{E} (V_{f,n}(\delta))^2 \leq \varepsilon^{-2} \delta^{-2\alpha} n^{-1} \mathbb{E} \max_{1 \leq \ell \leq \lfloor n\delta \rfloor} \left(\sum_{j=1}^{\ell} f \circ \mathbb{T}^j \right)^2 \\ &\leq \varepsilon^{-2} \delta^{1-2\alpha} \mathbb{E} f^2 \end{aligned}$$

and (4.1.4) follows. This ends the proof of (4.1.4) in the case $p = 2$.

Assume that $p > 2$ and $\alpha \leq 1/p$. Let us fix $\varepsilon > 0$. Define for any $\delta \in (0, 1)$ and any integer $n \geq 1/\delta$ the events

$$\begin{aligned} A_{n,\delta} &:= \left(\delta^{-\alpha} n^{-1/2} \max_{1 \leq \ell \leq \lfloor n\delta \rfloor} \left(\frac{1}{n} \sum_{k=0}^{n-\ell} \left| \sum_{j=k}^{k+\ell-1} f \circ \mathbb{T}^j \right|^p \right)^{1/p} > \varepsilon \right) \\ B_{n,\delta,\tau} &:= \left(\max_{1 \leq \ell \leq \lfloor n\delta \rfloor} n^{-1/2} \max_{1 \leq k \leq n-\ell} |S_{f,k+\ell} - S_{f,k}| \geq \varepsilon^{\frac{p}{p-2}} \tau^{\frac{p}{p-2}} \right), \end{aligned}$$

where τ is an arbitrary but fixed positive number. We have the bound

$$\mathbb{P}(B_{n,\delta,\tau}) \leq \mathbb{P} \left(\sup_{0 \leq s < t \leq 1, |t-s| < \delta} n^{-1/2} |\zeta_{f,n}(t) - \zeta_{f,n}(s)| \geq \varepsilon^{\frac{p}{p-2}} \tau^{\frac{p}{p-2}} \right).$$

Since the sequence $(\zeta_{f,n})$ is tight in the space $C[0, 1]$ (see [1]), we have

$$\lim_{\delta \rightarrow 0} \sup_{n \geq 1/\delta} \mathbb{P}(B_{n,\delta,\tau}) = 0. \quad (4.1.6)$$

Now, note that on $A_{n,j} \cap B_{n,j,\tau}^c$, we have for any $0 \leq \ell \leq \lfloor n\delta \rfloor$,

$$\begin{aligned} \sum_{k=0}^{n-\ell} \left| \sum_{j=k}^{k+\ell-1} f \circ \mathbb{T}^j \right|^p &\leq \sum_{k=0}^{n-\ell} \left| \sum_{j=k}^{k+\ell-1} f \circ \mathbb{T}^j \right|^2 \max_{1 \leq \ell \leq \lfloor n\delta \rfloor} \left(n^{-1/2} \max_{1 \leq k \leq n-\ell} |S_{f,k+\ell} - S_{f,k}| \right)^{p-2} n^{(p-2)/2} \\ &\leq \sum_{k=0}^{n-\ell} \left| \sum_{j=k}^{k+\ell-1} f \circ \mathbb{T}^j \right|^2 n^{(p-2)/2} \varepsilon^p \tau^p. \end{aligned} \quad (4.1.7)$$

$$\leq \sum_{k=0}^{n-\ell} \left| \sum_{j=k}^{k+\ell-1} f \circ \mathbb{T}^j \right|^2 n^{(p-2)/2} \varepsilon^p \tau^p. \quad (4.1.8)$$

Therefore, we have

$$\begin{aligned} \varepsilon &< \delta^{-\alpha} n^{-1/2} \max_{1 \leq \ell \leq \lfloor n\delta \rfloor} \left(\frac{1}{n} \sum_{k=0}^{n-\ell} \left| \sum_{j=k}^{k+\ell-1} f \circ \mathbb{T}^j \right|^p \right)^{1/p} \\ &\leq \delta^{-\alpha} n^{-1/2} n^{-1/p} \left(n^{(p-2)/2} \varepsilon^p \tau^p \right)^{1/p} \max_{1 \leq \ell \leq \lfloor n\delta \rfloor} \left(\sum_{k=0}^{n-\ell} \left| \sum_{j=k}^{k+\ell-1} f \circ \mathbb{T}^j \right|^2 \right)^{1/p} \\ &= \delta^{-\alpha} n^{-2/p} \varepsilon \tau \max_{1 \leq \ell \leq \lfloor n\delta \rfloor} \left(\sum_{k=0}^{n-\ell} \left| \sum_{j=k}^{k+\ell-1} f \circ \mathbb{T}^j \right|^2 \right)^{1/p}, \end{aligned}$$

and we infer that

$$\mathbb{P}(A_{n,\delta} \cap B_{n,\delta,\tau}^c) \leq \mathbb{P}\left(\delta^{-\alpha p} n^{-2} \tau^p \max_{1 \leq \ell \leq \lfloor n\delta \rfloor} \sum_{k=0}^{n-\ell} \left| \sum_{j=k}^{k+\ell-1} f \circ \mathbb{T}^j \right|^2 > 1\right).$$

By Markov's inequality, stationarity and Doob's inequality, we have

$$\begin{aligned} \mathbb{P}(A_{n,\delta} \cap B_{n,\delta,\tau}^c) &\leq \delta^{-\alpha p} n^{-2} \tau^p \mathbb{E} \left(\max_{1 \leq \ell \leq \lfloor n\delta \rfloor} \sum_{k=0}^{n-\ell} \left| \sum_{j=k}^{k+\ell-1} f \circ \mathbb{T}^j \right|^2 \right) \\ &\leq \delta^{-\alpha p} n^{-2} \tau^p \mathbb{E} \left(\sum_{k=0}^n \max_{1 \leq \ell \leq \lfloor n\delta \rfloor} \left| \sum_{j=k}^{k+\ell-1} f \circ \mathbb{T}^j \right|^2 \right) \\ &= \delta^{-\alpha p} n^{-1} \tau^p \mathbb{E} \left(\max_{1 \leq \ell \leq \lfloor n\delta \rfloor} |S_{f,\ell}|^2 \right) \\ &\leq 2\delta^{1-\alpha p} \tau^p \mathbb{E}(f^2). \end{aligned}$$

Since $p\alpha \leq 1$, we get

$$\mathbb{P}(A_{n,\delta} \cap B_{n,\delta}^c) \leq 2\tau^p \mathbb{E}(f^2)$$

and since τ is arbitrary, we get

$$\lim_{\delta \rightarrow 0} \sup_{n \geq 1/\delta} \mathbb{P}(A_{n,\delta}) = 0$$

in view of (4.1.6). This concludes the proof of (4.1.4) and that of Theorem 1.4.

4.2. Proof of Theorem 1.5. It follows from Theorem 1.4 and the same arguments as used in the proof of Theorem 1.2.

5. SOME APPLICATIONS

As already was mentioned in the introduction, a choice of functional spaces for polygonal line processes is usually inspired by possible applications in statistics via continuous mappings: if $W_{f,n} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} W$ in the space $B_{p,\alpha}^o$, then $T(W_{f,n}) \rightarrow T(W)$ for any continuous function $T : B_{p,\alpha}^o \rightarrow \mathbb{R}$. This general observation can be used, for example, to analyse so called k -scan processes

$$S_{f,k}^{(i)} = \sum_{j=i}^{i+k-1} f \circ \mathbb{T}^j, \quad i = 1, \dots, n-k+1.$$

Proposition 5.1. *Let f be a function such that the sequence $(W_n(f))_{n \geq 1}$ converges to a standard Brownian motion W in $B_{p,\alpha}^o$, where $1/p < \alpha < 1/2$. For each $a, b \in [0, 1]$ such that $a < b$, we define*

$$Y_{n,a,b}(f) := n^{-1/q(p,\alpha)} \max_{\lfloor an \rfloor < k \leq \lfloor bn \rfloor + 1} \frac{1}{k^\alpha} \left(\sum_{i=0}^{n-k} |S_{f,k}^{(i)}|^p \right)^{1/p}.$$

Then the following convergence holds:

$$Y_{n,a,b}(f) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Y_{a,b} := \sup_{a < t \leq b} t^{-\alpha} \left(\int_{I_t} |W(s+t) - W(s)|^p ds \right)^{1/p}. \quad (5.0.1)$$

Proof. Let us define a functional $F: B_{p,\alpha}^o \rightarrow \mathbb{R}$ by

$$F(x) := \sup_{a < t \leq b} t^{-\alpha} \left(\int_{I_t} |x(s+t) - x(s)|^p ds \right)^{1/p}.$$

Then F is continuous with respect to the topology of $B_{p,\alpha}^o$ and $F(W) = Y_{a,b}$. We thus have $F(W_n(f)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Y_{a,b}$. To conclude that (5.0.1) holds, it suffices to prove that

$$Z_n := F(W_n(f)) - Y_{n,a,b}(f) \rightarrow 0 \text{ in probability as } n \rightarrow +\infty.$$

First note that Z_n is non-negative. Second, we have

$$\begin{aligned} F(W_n(f)) &\leq \max_{[an] \leq k \leq [bn]+1} \max_{k/n \leq t < (k+1)/n} t^{-\alpha} \left(\int_{I_t} |W_n(f, s+t) - W_n(f, s)|^p ds \right)^{1/p} \\ &\leq \max_{[an] \leq k \leq [bn]+1} \max_{k/n \leq t < (k+1)/n} \left(\frac{k}{n} \right)^{-\alpha} \left(\int_{I_t} |W_n(f, s+t) - W_n(f, s)|^p ds \right)^{1/p}. \end{aligned}$$

Let k be an integer such that $[an] \leq k \leq [bn] + 1$ and let t be a real number such that $k/n \leq t < (k+1)/n$. Then

$$\begin{aligned} &\left| \left(\int_{I_t} |W_n(s+t) - W_n(s)|^p ds \right)^{1/p} - \left(\int_{I_{k/n}} |W_n(s+k/n) - W_n(s)|^p ds \right)^{1/p} \right| \\ &\leq \left(\int_{1-t}^{1-k/n} |W_n(f, s+k/n) - W_n(s)|^p ds \right)^{1/p} + \left(\int_0^{1-t} |W_n(f, s+t) - W_n(s+k/n)|^p ds \right)^{1/p} \\ &\leq \left(\int_{1-(k+1)/n}^{1-k/n} |W_n(f, s+k/n) - W_n(s)|^p ds \right)^{1/p} + \omega_p(W_n(f), t - k/n) \\ &= \left(\int_{1-(k+1)/n}^{1-k/n} |W_n(f, s+k/n) - W_n(s)|^p ds \right)^{1/p} + \omega_p(W_n(f), t - k/n), \end{aligned}$$

which implies that

$$\begin{aligned} F(W_n(f)) - Y_{n,a,b}(f) &\leq \max_{[an] \leq k \leq [bn]+1} \left(\frac{k}{n} \right)^{-\alpha} \left(\int_{1-(k+1)/n}^{1-k/n} |W_n(f, s+k/n) - W_n(s)|^p ds \right)^{1/p} \\ &\quad + \max_{[an] \leq k \leq [bn]+1} \left(\frac{1}{k} \right)^{\alpha} \sup_{|\delta| \leq 1/n} \delta^{-\alpha} \omega_p(W_n(f), \delta) \\ &\leq \max_{1 \leq k \leq n-1} \left(\frac{n}{k} \right)^{\alpha} \left(\int_{1-(k+1)/n}^{1-k/n} |W_n(f, s+k/n) - W_n(s)|^p ds \right)^{1/p} \\ &\quad + \sup_{0 < |\delta| \leq 1/n} \delta^{-\alpha} \omega_p(W_n(f), \delta). \quad (5.0.2) \end{aligned}$$

By Theorem 2.4, the second term in the right hand side of (5.0.2) goes to zero in probability. Therefore, it suffices to prove that

$$\max_{1 \leq k \leq n-1} \left(\frac{n}{k} \right)^\alpha \left(\int_{1-\frac{k+1}{n}}^{1-\frac{k}{n}} \left| W_n \left(f, s + \frac{k}{n} \right) - W_n(f, s) \right|^p ds \right)^{\frac{1}{p}} \rightarrow 0 \text{ in probability as } n \rightarrow +\infty. \quad (5.0.3)$$

To see this, we start from the inequality

$$\begin{aligned} \max_{1 \leq k \leq n-1} \left(\frac{n}{k} \right)^\alpha \left(\int_{1-(k+1)/n}^{1-k/n} \left| W_n \left(f, s + \frac{k}{n} \right) - W_n(f, s) \right|^p ds \right)^{1/p} \\ \leq \sup_{0 < t < 1} t^{-\alpha} \left(\int_{1-t-1/n}^{1-t} |W_n(f, s+t) - W_n(f, s)|^p ds \right)^{1/p} \\ = \sup_{0 < t < 1} t^{-\alpha} \left(\int_{1-1/n}^1 |W_n(f, s) - W_n(f, s-t)|^p ds \right)^{1/p}. \end{aligned}$$

Let N be a fixed integer. The functional

$$G_N : B_{p,\alpha}^o \rightarrow \mathbb{R}, \quad G_N(x) = \sup_{0 < t < 1} t^{-\alpha} \left(\int_{[1-1/N, 1] \cap I_{-t}} |x(s) - x(s-t)|^p ds \right)^{1/p}$$

is continuous. Therefore, if $\varepsilon > 0$ is a continuity point of the cumulative distribution function of $G_N(W)$ for each N , we have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \mathbb{P} \left(\max_{1 \leq k \leq n-1} \left(\frac{n}{k} \right)^\alpha \left(\int_{1-(k+1)/n}^{1-k/n} |W_n(f, s+k/n) - W_n(f, s)|^p ds \right)^{1/p} > \varepsilon \right) \\ \leq \mathbb{P} \left(\sup_{0 < t < 1} t^{-\alpha} \left(\int_{[1-1/N, 1] \cap I_{-t}} |W(s) - W(s-t)|^p ds \right)^{1/p} > \varepsilon \right). \quad (5.0.4) \end{aligned}$$

Now, take $q > p$. Using Hölder's inequality with the exponents q/p and $q/(q-p)$, we have

$$\begin{aligned} \sup_{0 < t < 1} t^{-\alpha} \left(\int_{[1-1/N, 1] \cap I_{-t}} |W(s) - W(s-t)|^p ds \right)^{1/p} \\ \leq \sup_{0 < t < 1} t^{-\alpha} \left(\int_{I_{-t}} |W(s) - W(s-t)|^q ds \right)^{1/q} N^{-(p-q)/q} \leq \|W\|_{q,\alpha} N^{-(p-q)/q}. \end{aligned}$$

Since $\|W\|_{q,\alpha}$ is almost surely finite, we get from (5.0.4) that (5.0.3) holds. \square

Statistics based on k -scan processes can be used to detect epidemic change in the mean of a sample of size n (see, e.g., [11], and reference therein). More precisely, given a sample X_1, X_2, \dots, X_n , consider the model

$$X_i = \mu \mathbf{1}_{(k^*, m^*]}(i) + \varepsilon_i, \quad i = 1, \dots, n,$$

where (ε_i) is a stationary sequence, $\mu \neq 0$ and k^*, m^* are unknown parameters of the model. We want to test the null hypothesis $H_0 : \mu = 0$ against the alternative $\mu \neq 0$. To this aim

one can consider the statistics

$$T_n = n^{-1/q(p,\alpha)} \max_{0 < k \leq n} \frac{1}{k^\alpha} \left(\sum_{i=0}^{n-k} |X_i + \dots + X_{i+k}|^p \right)^{1/p}.$$

Under null its limit is defined by Proposition 5.1 provided (ε_i) satisfies the weak invariance principle in the Besov space $B_{p,\alpha}^o$. Under alternative then we see that

$$T_n = \left(1 - \frac{h^*}{n}\right)^{1/p} \left(\frac{h^*}{n}\right)^{1-\alpha} n^{1/2} + O_P(1)$$

as $n \rightarrow \infty$, where $h^* = m^* - k^*$ is the duration of the epidemic state.

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NORMANDIE UNIV, UNIROUEN, CNRS, LMRS, 76000 ROUEN, FRANCE, DEPARTMENT OF MATHEMATICS AND INFORMATICS, VILNIUS UNIVERSITY, NAUGARDUKO 24, LT-03225 VILNIUS, LITHUANIA,
E-mail address: `davide.giraudo1@univ-rouen.fr`, `alfredas.rackauskas@mif.vu.lt`.